LECTURE 20

ABSTRACT DATA TYPES
AND
DATA STRUCTURES

- An abstract data type
  - Specifies supported operations

- A data structure is an implementation of an abstract data type
  - Specifies data layout in memory
  - Specifies algorithms for operations

Lists

- support storing multiple elements together

  and optionally append, insert, delete, random access, …
- implementations:
  - dynamic arrays

    everything $O(1)$ in practice except insert/delete $O(n)$
  - linked lists

    everything $O(1)$ (but slower than arrays) except random access $O(n)$

STACKS / LIFO

- A stack is an ordered collection of elements
- supports two operations:
    - “push”: add an element
    - “pop”: retrieve-and-remove the last-added element

    $\Rightarrow$ last in, first out (LIFO)

Static array implementation of a stack

- Useful only when there is a hard limit on the number of elements
- We maintain a static array
- and a **stack pointer** (or **top index**)
- this is how “the” stack is implemented

  (for storing function arguments, local variables and return addresses)

→ push A

push B

push C

pop

pop

push E

push F

push G

pop

pop

pop

pop

.

A	x	x	x	x	x	x	x	x	x	x	x

↑ stack pointer

push A

→ push B

push C

pop

pop

push E

push F

push G

pop

pop

pop

pop

.

push A

push B

→ push C

pop

pop

push E

push F

push G

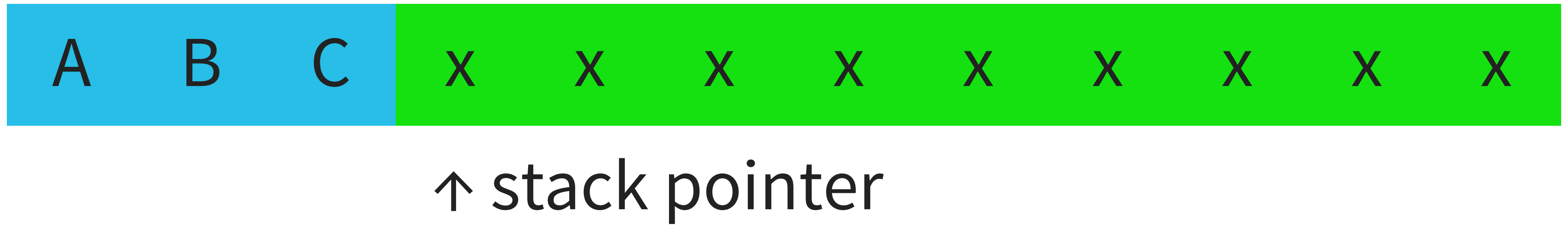
pop

pop

pop

pop

.



push A
push B
push C
→ pop
pop
push E
push F
push G
pop
pop
pop
pop
.

push A
push B
push C
pop → **C**
→ pop
push E
push F
push G
pop
pop
pop
pop
.

push A

push B

push C

pop → **C**

pop → **B**

→ push E

push F

push G

pop

pop

pop

pop

.

A	E	C	x	x	x	x	x	x	x	x	x

↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

→ push F

push G

pop

pop

pop

pop

.

A	E	F	x	x	x	x	x	x	x	x	x

↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

→ push G

pop

pop

pop

pop

.

A	E	F	G	x	x	x	x	x	x	x	x

↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

→ pop

pop

pop

pop

.

A	E	F	G	x	x	x	x	x	x	x	x

↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

pop → **G**

→ pop

pop

pop

.

A	E	F	G	x	x	x	x	x	x	x	x

↑ stack pointer

push A
push B
push C
pop → **C**
pop → **B**
push E
push F
push G
pop → **G**
pop → **F**
→ pop
pop
.

A	E	F	G	x	x	x	x	x	x	x	x

↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

pop → **G**

pop → **F**

pop → **E**

→ pop

.

A	E	F	G	x	x	x	x	x	x	x	x

↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

pop → **G**

pop → **F**

pop → **E**

pop → **A**

→ .

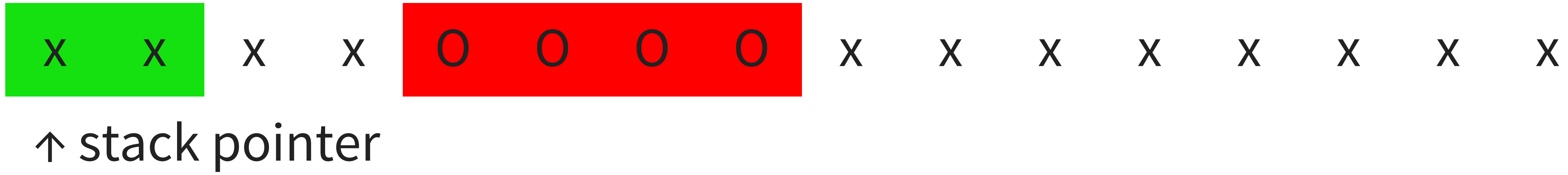
Linked list implementation of a stack

- Pro: No hard limit on number of elements

- Con:
  - Memory allocation for every **push**
  - Memory freed for every **pop**

Dynamic array implementation of a stack

- Pros:
  - No hard limit on number of elements
  - Memory management overhead is small
- Con:
  - No pointer stability



→ push A

push B

push C

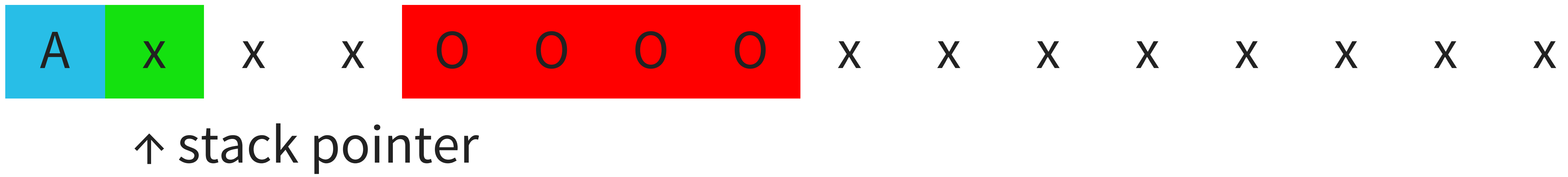
p = address of C

push D

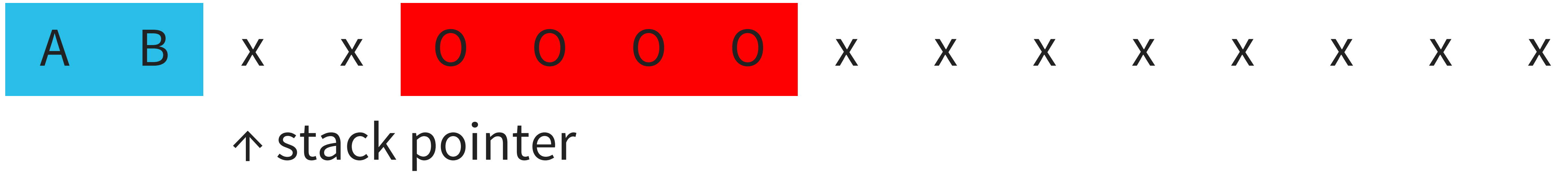
push E

change C into C' using p

.




```
  push A
→ push B
  push C
  p = address of C
  push D
  push E
  change C into C’ using p
  .
```



push A

push B

→ push C

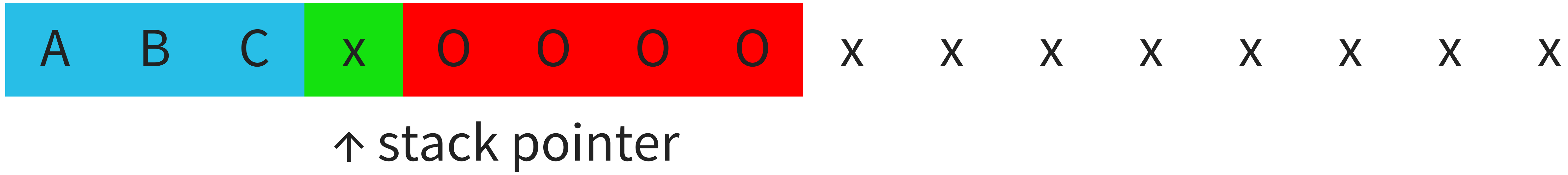
p = address of C

push D

push E

change C into C' using p

.



push A

push B

push C

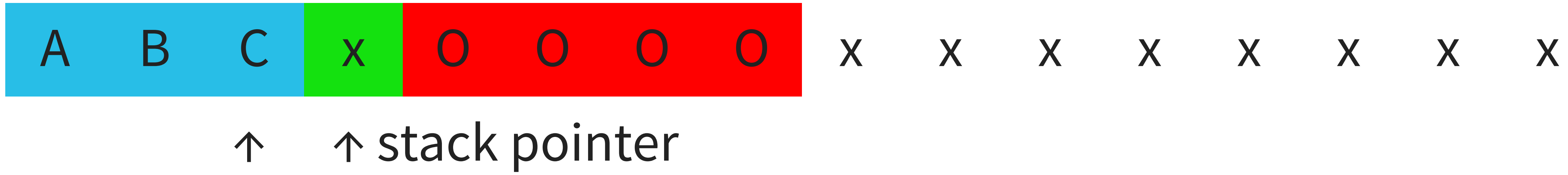
→ p = address of C

push D

push E

change C into C’ using p

.



push A

push B

push C

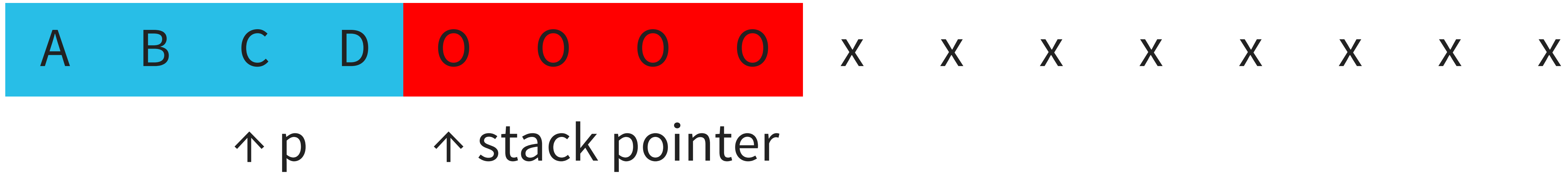
p = address of C

→ push D

push E

change C into C’ using p

.



push A

push B

push C

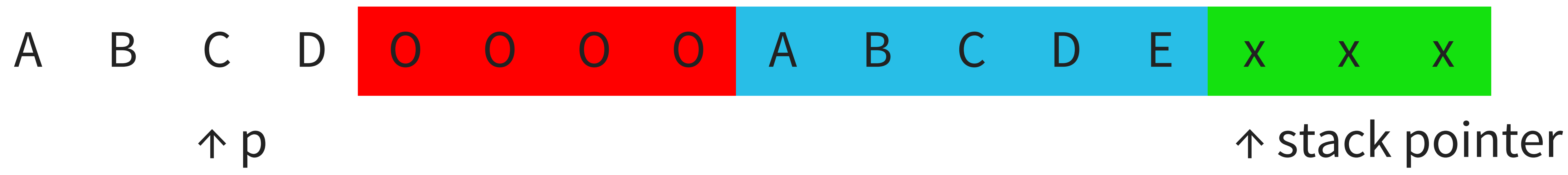
p = address of C

push D

→ push E

change C into C’ using p

.



push A

push B

push C

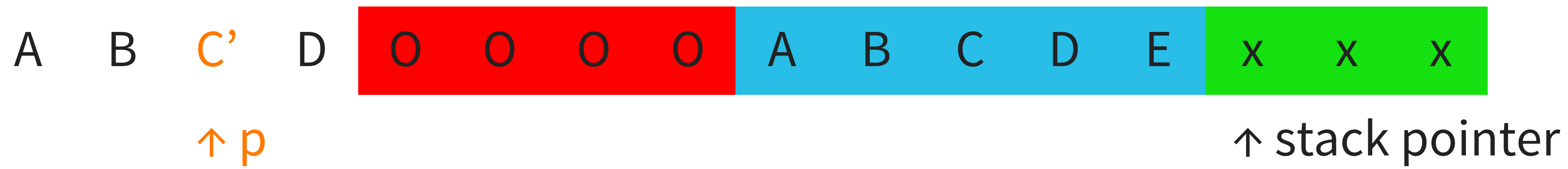
p = address of C

push D

push E

→ change C into C’ using p

.



push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

→ .

Stack implementations

Implementation	Size	Requires allocations	Pointer stability
static array	constant	no	yes
dynamic array	can grow	when growing	no
linked list	can grow	every push	yes

Arena

- Known as arena allocator, region-based allocator, zone-based allocator, obstack
- Implemented as a list of static array stacks

x	x	x	x	O	O	O	O	x	x	x	x	x	x	x	x

↑ stack pointer

↑ block 0

→ push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop

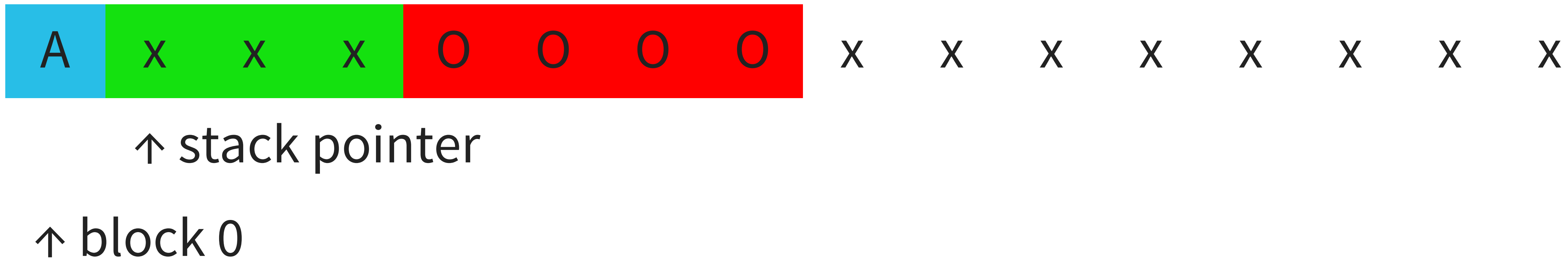
pop

pop

pop

.

list of regions: block 0



push A

→ push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop

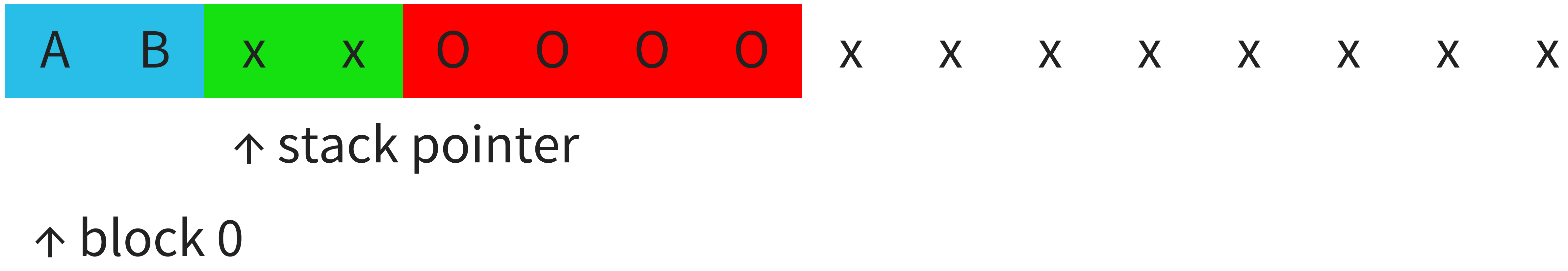
pop

pop

pop

.

list of regions: block 0



push A

push B

→ push C

p = address of C

push D

push E

change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0

A B C x O O O O x x x x x x x x

↑ stack pointer

↑ block 0

push A

push B

push C

→ p = address of C

push D

push E

change C into C’ using p

push F

pop

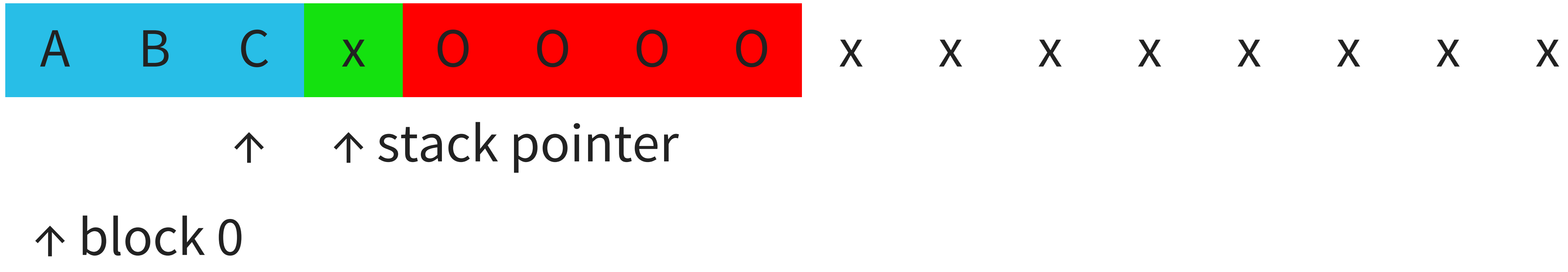
pop

pop

pop

.

list of regions: block 0



push A

push B

push C

p = address of C

→ push D

push E

change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0

A	B	C	D	O	O	O	O	x	x	x	x	x	x	x	x

↑ p ↑ stack pointer

↑ block 0

push A

push B

push C

p = address of C

push D

→ push E

change C into C' using p

push F

pop

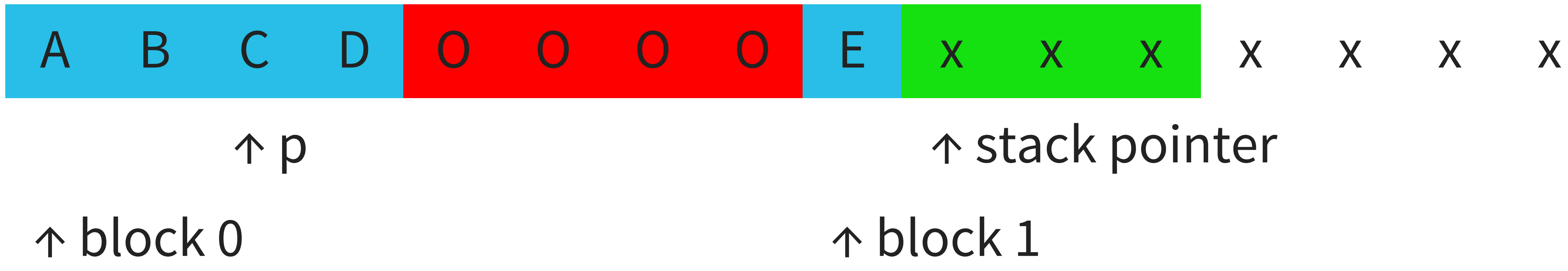
pop

pop

pop

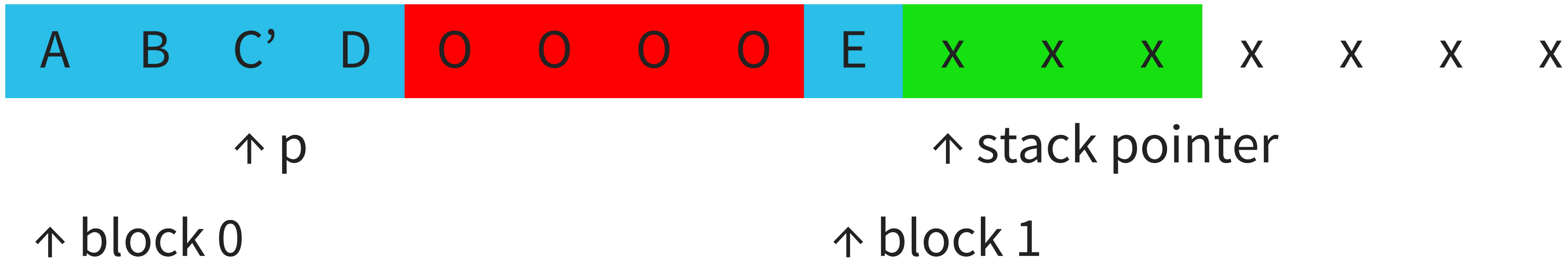
.

list of regions: block 0



push A
push B
push C
p = address of C
push D
push E
→ change C into C’ using p
push F
pop
pop
pop
pop
.

list of regions: block 0
block 1



push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

→ push F

pop

pop

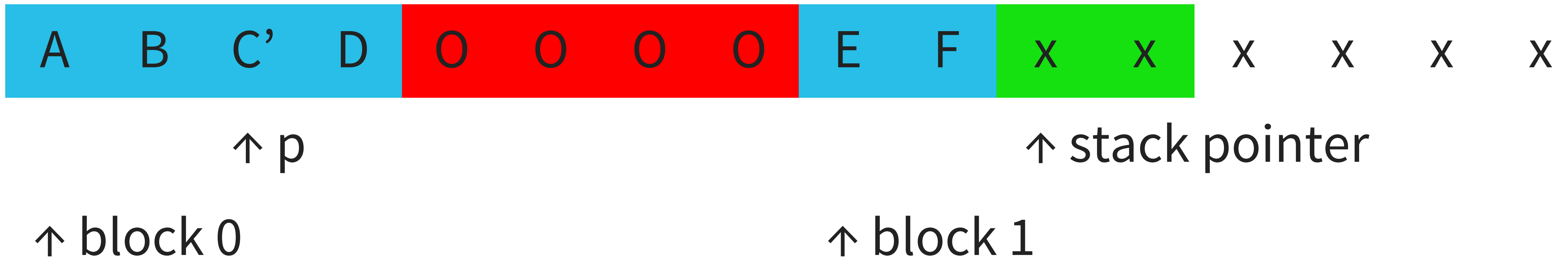
pop

pop

.

list of regions: block 0

block 1



push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

→ pop

pop

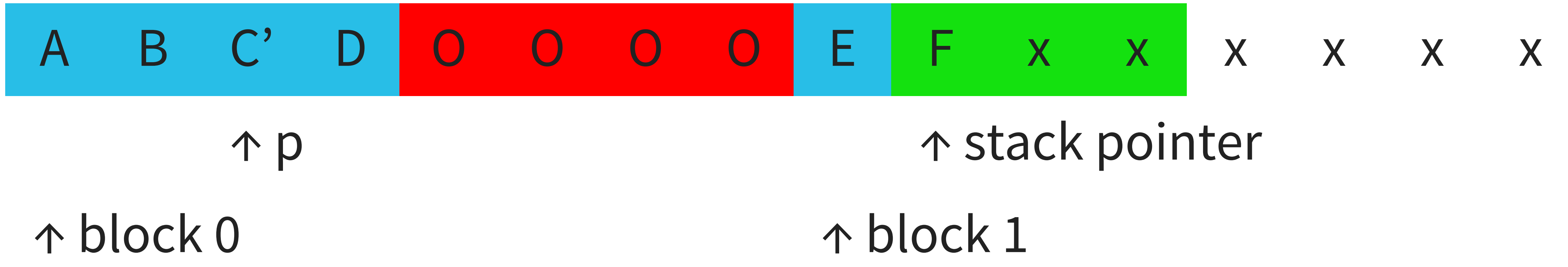
pop

pop

.

list of regions: block 0

block 1



push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop → F

→ pop

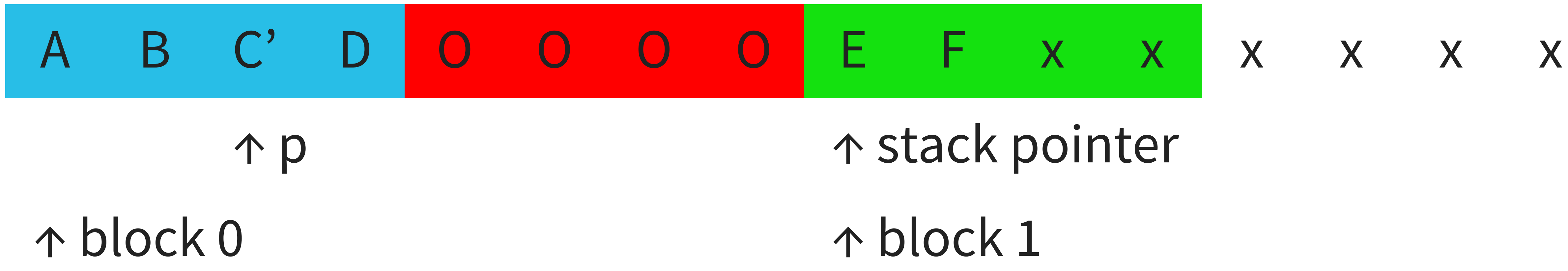
pop

pop

.

list of regions: block 0

block 1



push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop → F

pop → E

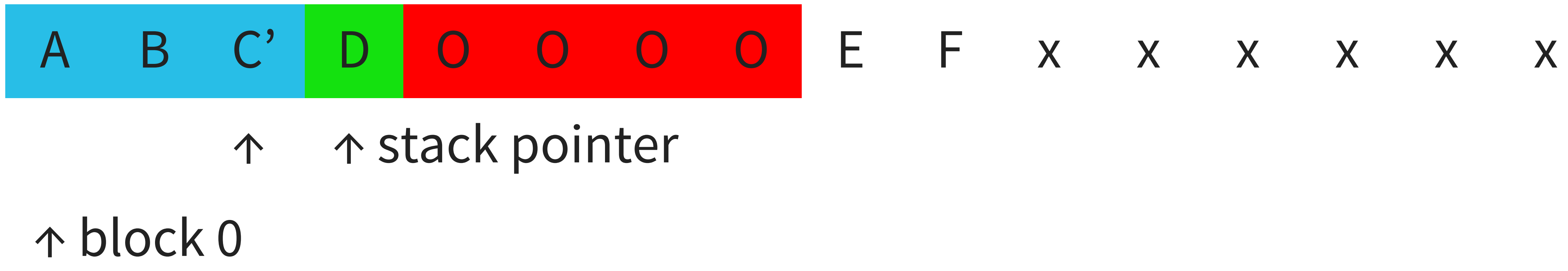
→ pop

pop

.

list of regions: block 0

block 1



push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop → F

pop → E

pop → D

→ pop

.

list of regions: block 0

A B C’ D O O O O E F x x x x x x

↑ stack pointer

↑ block 0

push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop → F

pop → E

pop → D

pop → C’

→ .

list of regions: block 0

Stack implementations

Implementation	Size	Requires allocations	Pointer stability
static array	constant	no	yes
dynamic array	can grow	when growing	no
linked list	can grow	every push	yes
arena	can grow	when growing	yes

More options

Combination of other data structures and indirection can be used, depending on desired properties.

QUEUES / FIFO

- A queue is an ordered collection of elements
- supports two operations:
  - “enqueue”: add an element
  - “dequeue”: retrieve-and-remove the earliest-added element

  $\Rightarrow$ first in, first out (FIFO)

Ring buffer implementation of a queue

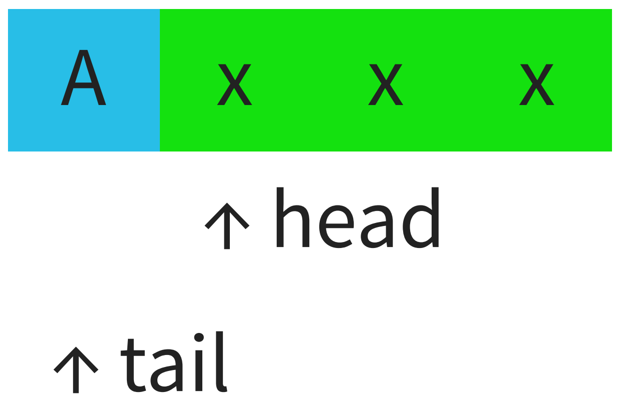
- Useful only when there is a hard limit on the number of elements
- We maintain a static array
- and two pointers/indices: **head** and **tail**

x	x	x	x

↑ head

↑ tail

→ enqueue A
enqueue B
dequeue
dequeue
enqueue C
enqueue D
enqueue E
enqueue F
dequeue
dequeue
dequeue
enqueue G
dequeue
dequeue
.



enqueue A

→ enqueue B

dequeue

dequeue

enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

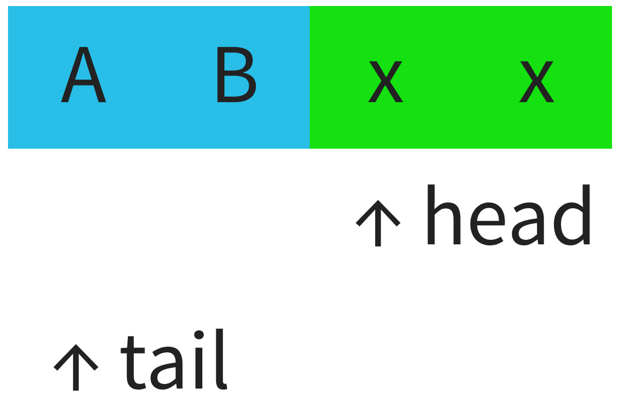
dequeue

enqueue G

dequeue

dequeue

.



enqueue A

enqueue B

→ dequeue

dequeue

enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

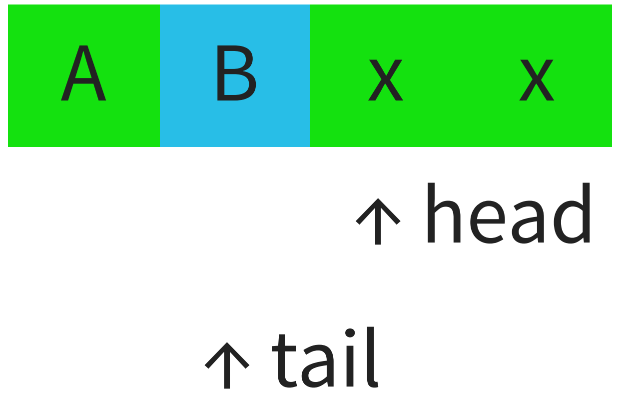
dequeue

enqueue G

dequeue

dequeue

.



enqueue A

enqueue B

dequeue → A

→ dequeue

enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

dequeue

enqueue G

dequeue

dequeue

.

A	B	x	x

↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

→ enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

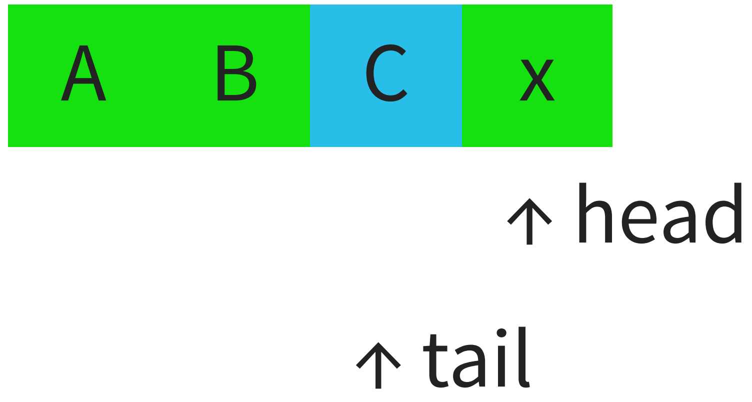
dequeue

enqueue G

dequeue

dequeue

.



enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

→ enqueue D

enqueue E

enqueue F

dequeue

dequeue

dequeue

enqueue G

dequeue

dequeue

.

A	B	C	D

↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

→ enqueue E

enqueue F

dequeue

dequeue

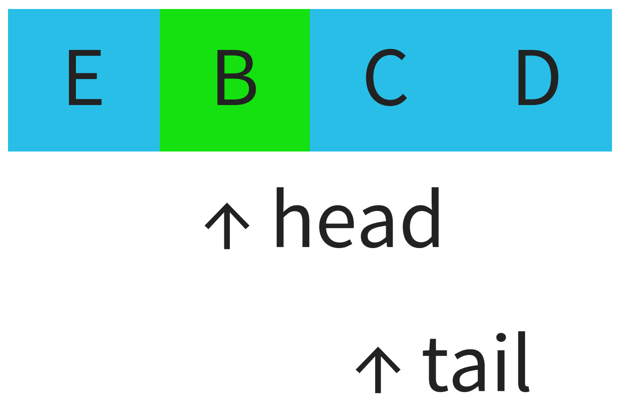
dequeue

enqueue G

dequeue

dequeue

.



enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

→ enqueue F

dequeue

dequeue

dequeue

enqueue G

dequeue

dequeue

.

E	F	C	D

↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

→ dequeue

dequeue

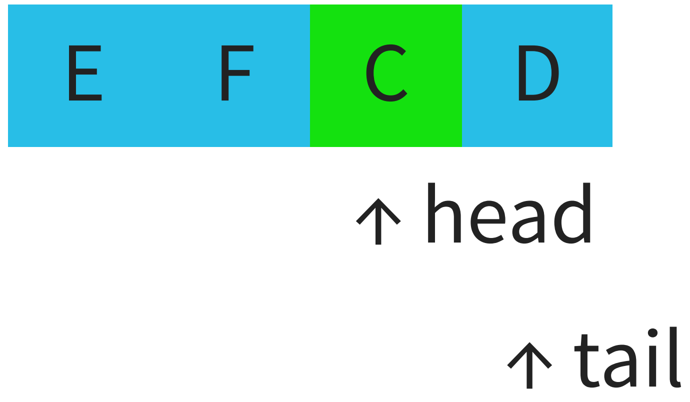
dequeue

enqueue G

dequeue

dequeue

.



enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

→ dequeue

dequeue

enqueue G

dequeue

dequeue

.

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

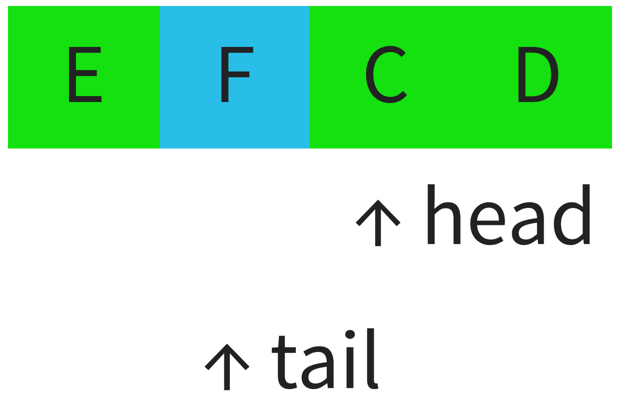
→ dequeue

enqueue G

dequeue

dequeue

.



enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

dequeue → E

→ enqueue G

dequeue

dequeue

.

E	F	G	D

↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

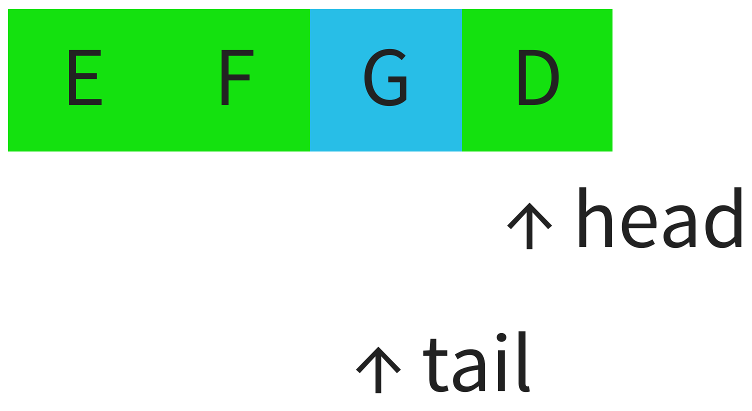
dequeue → E

enqueue G

→ dequeue

dequeue

.



enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

dequeue → E

enqueue G

dequeue → F

→ dequeue

.

E	F	G	D

↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

dequeue → E

enqueue G

dequeue → F

dequeue → G

→ .

Implementation detail

When head == tail:

the queue is empty?

E	F	G	D

↑ head

↑ tail

or the queue is full?

E	F	C	D

↑ head

↑ tail

- maintain a variable with the number of elements currently in the queue
- or keep incrementing head and tail, and index the static array as

  `array[head % size]` and

  `array[tail % size]`

Applications of ring buffers

- Audio playback/recording devices
- Video capture devices
- Special case (double-buffering, i.e. size = 2) for computer graphics
- Network devices (routers, switches)

More options

- use dynamic arrays
- use linked lists
- use indirection
- …
- depending on specific needs

PRIORITY QUEUES

- A priority queue is a collection of elements, each with an associated priority
- supports two operations:
  - “push”: add an element-priority tuple
  - “pop”: retrieve-and-remove the highest-priority element

Implementation of a priority queue

- Store element-priority tuples in an array or in a linked list
- “push”: $O(1)$ of the underlying data structure
- “pop”: scan all elements, find max priority, $O(n)$

Binary heap implementation of a priority queue

```
           30
        /      \
     25          15
    /  \        /
   5    18     9
```

- Binary heaps represent a priority queue as
  - a binary tree (every node has at most two children)
  - that is complete (every level full, except possibly the deepest)
- Every node is labeled by the corresponding element's priority
- Tree has the heap property:
  - Priority of any node $\geq$ priority of its children
  - $\Leftrightarrow$ Priority of any node $\geq$ priority of all its descendants

Binary heap push

```
            30
        /       \
    25             15
  /   \          /   \
 5    18        9     x
```

- Step 0: Add new element at the first free slot on the deepest level
- Step 1:
  - If its priority is not higher than its parent's,
    - the heap property is satisfied, we are done
  - If its priority is higher than its parent's,
    - swap them,
    - go back to Step 1, looking at the pushed element's new position

Binary heap push

```
           30
       /       \
    25           x
   /  \        /   \
  5    18     9     15
```

- Step 0: Add new element at the first free slot on the deepest level
- Step 1:
  - If its priority is not higher than its parent's,
    - the heap property is satisfied, we are done
  - If its priority is higher than its parent's,
    - swap them,
    - go back to Step 1, looking at the pushed element's new position

Binary heap push

```
          x
       /      \
   25            30
  /  \          /  \
 5    18       9     15
```

- Step 0: Add new element at the first free slot on the deepest level
- Step 1:
  - If its priority is not higher than its parent's,
    - the heap property is satisfied, we are done
  - If its priority is higher than its parent's,
    - swap them,
    - go back to Step 1, looking at the pushed element's new position

Binary heap push: 14

```
          30
       /       \
    25           15
   /  \         /   \
  5    18      9     14    <---  heap property satisfied
```

- Add new element at the first free slot on the deepest level
- If it is not higher than its parent's,
    - the heap property is satisfied, we are done

Binary heap push: 28

```
              30
           /      \
        25           15
       /  \         /  \
      5    18      9    28    <---  heap property NOT satisfied
```

- Add new element at the first free slot on the deepest level
- If its priority is higher than its parent's,
  - swap them

Binary heap push: 28

```
          30
        /      \
     25          28   <---  heap property satisfied below this (28 > 15)
    /  \        /  \
   5   18      9    15
```

- Heap property satisfied below the pushed element’s new position
- If its priority is not higher than its new parent’s,
  - the heap property is satisfied, we are done

Binary heap push: 35

```
              30
          /        \
       25              15
     /   \            /   \
    5    18         9     35    <---  heap property NOT satisfied
```

- Add new element at the first free slot on the deepest level
- If its priority is higher than its parent's,
  - swap them

Binary heap push: 35

```
           30
         /      \
      25          35      <--- heap property NOT satisfied above
     /  \        /  \
    5   18      9    15
```

- Heap property satisfied below the pushed element's new position
- If its priority is still higher than its new parent's,
    - swap them

Binary heap push: 35

```
             35
          /       \
      25            30    <---  heap property satisfied below this (30 > 15)
     /   \         /   \
    5    18       9     15
```

- Heap property satisfied below the pushed element’s new position
  - new child was an ancestor of its direct children
- Continue until heap property is satisfied

Binary heap pop

```
          30
       /       \
    25           15
   /  \         /
  5    18       9
```

- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

Binary heap pop

```
          9
       /     \
    25          15
   /  \
  5    18
```

- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

Binary heap pop

```
           25         <--- priority ≥ its children's
         /       \
       9           15      <--- heap property untouched here
     /   \
   5    18
```

- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

Binary heap pop

```
            25
          /       \
       18            15                                   18 was a descendant of 25
      /   \
    5     9
```

- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

Binary heap operations

- Push: $O(\log_2(n))$
- Find max: $O(1)$
- Pop: $O(\log_2(n))$

Complete binary data structure

- Binary heaps are complete binary trees
- We can avoid allocation for every “push” by storing nodes in an array
- Depth ℓ of the tree has at most 2^ℓ nodes, $\forall \ell$
- Depth ℓ of the tree has exactly 2^ℓ nodes, except for the deepest level

depth	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

- There are exactly $(2^\ell - 1)$ nodes of with depth $< \ell$

Storage scheme

```
depth 0                                                 0
depth 1                                1                                       2
depth 2                          3            4                          5           6
depth 3                        7   8         9   10                   11   12       13   14
```

depth	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

- If a node has index j
- its children are stored at indices $2j+1$ and $2j+2$
- its parent is stored at index $\lfloor (j-1)/2 \rfloor$

Storage scheme

```
depth 0                                             0
depth 1                         1                                       2
depth 2                   3            4                       5              6
depth 3                 7   8        9   10                 11   12        13   14
```

depth	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

- If a node has index j
- its children are stored at indices $2j + 1$ and $2j + 2$
- its parent is stored at index $\lfloor (j - 1)/2 \rfloor$

Storage scheme

```
depth 0                                         0
depth 1                           1                                       2
depth 2                      3           4                        5           6
depth 3                    7    8       9   10                 11   12      13   14
```

depth	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

- If a node has index j
- its children are stored at indices $2j+1$ and $2j+2$
- its parent is stored at index $\lfloor (j-1)/2 \rfloor$

Storage scheme

```
depth 0                                             0
depth 1                           1                                       2
depth 2                       3               4                       5               6
depth 3                     7   8           9   10                  11   12          13   14
```

depth	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

- If a node has index j
- its children are stored at indices $2j+1$ and $2j+2$
- its parent is stored at index $\lfloor (j-1)/2 \rfloor$

Storage scheme

```
depth 0                                                 0
depth 1                                   1                                       2
depth 2                            3             4                         5             6
depth 3                          7   8         9   10                   11   12       13   14
```

depth	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

- If a node has index j
- its children are stored at indices $2j+1$ and $2j+2$
- its parent is stored at index $\lfloor (j-1)/2 \rfloor$

Binary heap with array storage

- Superior to in-memory tree (with pointers)
  - We avoid allocation for every “push”
  - We avoid data dependencies (load node data to get pointer to parent/children)
- Still,
  - Push and pop operations are tough for branch predictor
  - Jumps to indices $(2j+1)$, $(2j+2)$ or $\lfloor (j-1)/2 \rfloor$ not cache-friendly for large j

Priority queue: special case

- Assume that
  - priorities are distinct integers $p \in \{0, \ldots, P-1\}$
  - we always push at a priority $\leq$ current max priority
- Then,
  - we allocate a static array of size P
  - Push: store in array at index p — $O(1)$
  - Pop: sweep array backwards — $O(P/n)$ amortized

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
element																
max																↑

→ push A, 12

push B, 5

push C, 9

pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

pop

push H, 1

pop

pop

.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
element													A			
max																↑

push A, 12

→ push B, 5

push C, 9

pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

pop

push H, 1

pop

pop

.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
element						B							A			
max																↑

push A, 12

push B, 5

→ push C, 9

pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

pop

push H, 1

pop

pop

.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
element						B				C			A			
max																↑

push A, 12

push B, 5

push C, 9

→ pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

pop

push H, 1

pop

pop

.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element						B				C			A		
max													↑		

push A, 12
push B, 5
push C, 9
pop → A
→ push E, 3
push F, 10
pop
push G, 4
pop
pop
pop
push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element				E		B				C			A		
max													↑		

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
→ push F, 10
pop
push G, 4
pop
pop
pop
push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element				E		B				C	F		A		
max													↑		

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
→ pop
push G, 4
pop
pop
pop
push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element				E		B				C	F		A		
max											↑				

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
→ push G, 4
pop
pop
pop
push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element				E	G	B				C	F		A		
max											↑				

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
→ pop
pop
pop
push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element				E	G	B				C	F		A		
max										↑					

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
pop → C
→ pop
pop
push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element				E	G	B				C	F		A		
max						↑									

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
pop → C
pop → B
→ pop
push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element				E	G	B				C	F		A		
max					↑										

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
pop → C
pop → B
pop → G
→ push H, 1
pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element		H		E	G	B				C	F		A		
max					↑										

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
pop → C
pop → B
pop → G
push H, 1
→ pop
pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element		H		E	G	B				C	F		A		
max				↑											

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
pop → C
pop → B
pop → G
push H, 1
pop → E
→ pop
.

priority	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
element		H		E	G	B				C	F		A		
max		↑													

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
pop → C
pop → B
pop → G
push H, 1
pop → E
pop → H
→ .

Implementation details

- good for branch predictor
- great for caches
- we can store, additionally, an array of P bits ("bitmap")
    - bit p set to one if there is an element with priority p
    - makes "pop" operations essentially 64x faster

Applications of **bitmap priority queues**

- Linux kernel: scheduling parallel tasks
- Linear algebra: sparse matrices

SORT OPERATIONS

Heap sort

- Push n elements to heap: $O(n \log n)$
- Pop n elements one by one: $O(n \log n)$

$$\Rightarrow \quad O(n \log n) \text{ worst case}$$

Comparison sort methods

Method	Average	Worst case	Additional storage	Combines with insert. sort
Quicksort	$O(n\log(n))$	$O(n^2)$	none	yes
Merge sort	$O(n\log(n))$	$O(n\log(n))$	n	yes
Heap sort	$O(n\log(n))$	$O(n\log(n))$	none	no

Special case 1

- Assume that
  - we sort n elements with priorities $S \subseteq \{0, \ldots, P-1\}$
  - no two elements have the same priority (hence $P \geq n$)
- Then,
  - we represent the elements as a bitmap priority queue
  - Push: $O(n)$
  - Pop: $O(P)$

$$\Rightarrow \quad O(n+P)$$

Special case 2: **counting sort**

- Assume that
  - we sort n elements with priorities $S \subseteq \{0, \ldots, P-1\}$
  - $P \leq n$ (we may have duplicates)
- Then,
  - we allocate a static array `count` of size P
  - we allocate a static array `result` of size n
  - we count the number of occurences of each priority: $O(n)$
  - we sweep `count` backwards to determine offsets: $O(P) = O(n)$
  - we construct the sorted `result` list: $O(n)$

$$\Rightarrow \quad O(n)$$

Count

priority	3	6	1	3	3	7	5	6	1	6
↑										
count index	0	1	2	3	4	5	6	7		
count value	0	0	0	0	0	0	0	0		
↑										
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
	↑									
count index	0	1	2	3	4	5	6	7		
count value	0	0	0	1	0	0	0	0		
				↑						
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
		↑								
count index	0	1	2	3	4	5	6	7		
count value	0	0	0	1	0	0	1	0		
							↑			
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
			↑							
count index	0	1	2	3	4	5	6	7		
count value	0	1	0	1	0	0	1	0		
		↑								
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
				↑						
count index	0	1	2	3	4	5	6	7		
count value	0	1	0	2	0	0	1	0		
				↑						
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
					↑					
count index	0	1	2	3	4	5	6	7		
count value	0	1	0	3	0	0	1	0		
				↑						
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
						↑				
count index	0	1	2	3	4	5	6	7		
count value	0	1	0	3	0	0	1	1		
								↑		
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
							↑			
count index	0	1	2	3	4	5	6	7		
count value	0	1	0	3	0	1	1	1		
						↑				
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
								↑		
count index	0	1	2	3	4	5	6	7		
count value	0	1	0	3	0	1	2	1		
							↑			
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
									↑	
count index	0	1	2	3	4	5	6	7		
count value	0	2	0	3	0	1	2	1		
		↑								
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Count

priority	3	6	1	3	3	7	5	6	1	6
										↑
count index	0	1	2	3	4	5	6	7		
count value	0	2	0	3	0	1	3	1		
							↑			
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6	
											↑
count index	0	1	2	3	4	5	6	7			
count value	0	2	0	3	0	1	3	1			
								↑			
result index	0	1	2	3	4	5	6	7	8	9	
result											
↑											

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6	
											↑
count index	0	1	2	3	4	5	6	7			
count value	0	2	0	3	0	1	4	1			
							↑				
result index	0	1	2	3	4	5	6	7	8	9	
result											
↑											

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6
										↑
count index	0	1	2	3	4	5	6	7		
count value	0	2	0	3	0	5	4	1		
						↑				
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6
										↑
count index	0	1	2	3	4	5	6	7		
count value	0	2	0	3	5	5	4	1		
					↑					
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6	
											↑
count index	0	1	2	3	4	5	6	7			
count value	0	2	0	8	5	5	4	1			
				↑							
result index	0	1	2	3	4	5	6	7	8	9	
result											
↑											

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6	
											↑
count index	0	1	2	3	4	5	6	7			
count value	0	2	8	8	5	5	4	1			
			↑								
result index	0	1	2	3	4	5	6	7	8	9	
result											
↑											

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6	
											↑
count index	0	1	2	3	4	5	6	7			
count value	0	10	8	8	5	5	4	1			
		↑									
result index	0	1	2	3	4	5	6	7	8	9	
result											
↑											

Determine offsets

priority	3	6	1	3	3	7	5	6	1	6	
											↑
count index	0	1	2	3	4	5	6	7			
count value	10	10	8	8	5	5	4	1			
	↑										
result index	0	1	2	3	4	5	6	7	8	9	
result											
↑											

Construct result

priority	3	6	1	3	3	7	5	6	1	6
↑										
count index	0	1	2	3	4	5	6	7		
count value	10	10	8	8	5	5	4	1		
↑										
result index	0	1	2	3	4	5	6	7	8	9
result										
↑										

Construct result

priority	3	6	1	3	3	7	5	6	1	6
	↑									
count index	0	1	2	3	4	5	6	7		
count value	10	10	8	7	5	5	4	1		
				↑						
result index	0	1	2	3	4	5	6	7	8	9
result								3		
								↑		

Construct result

priority	3	6	1	3	3	7	5	6	1	6
		↑								
count index	0	1	2	3	4	5	6	7		
count value	10	10	8	7	5	5	3	1		
							↑			
result index	0	1	2	3	4	5	6	7	8	9
result				6				3		
				↑						

Construct result

priority	3	6	1	3	3	7	5	6	1	6
			↑							
count index	0	1	2	3	4	5	6	7		
count value	10	9	8	7	5	5	3	1		
		↑								
result index	0	1	2	3	4	5	6	7	8	9
result				6				3		1
										↑

Construct result

priority	3	6	1	3	3	7	5	6	1	6
				↑						
count index	0	1	2	3	4	5	6	7		
count value	10	9	8	6	5	5	3	1		
				↑						
result index	0	1	2	3	4	5	6	7	8	9
result				6			3	3		1
							↑			

Construct result

priority	3	6	1	3	3	7	5	6	1	6
					↑					
count index	0	1	2	3	4	5	6	7		
count value	10	9	8	5	5	5	3	1		
				↑						
result index	0	1	2	3	4	5	6	7	8	9
result				6		3	3	3		1
						↑				

Construct result

priority	3	6	1	3	3	7	5	6	1	6
						↑				
count index	0	1	2	3	4	5	6	7		
count value	10	9	8	5	5	5	3	0		
								↑		
result index	0	1	2	3	4	5	6	7	8	9
result	7			6		3	3	3		1
	↑									

Construct result

priority	3	6	1	3	3	7	5	6	1	6
							↑			
count index	0	1	2	3	4	5	6	7		
count value	10	9	8	5	5	4	3	0		
						↑				
result index	0	1	2	3	4	5	6	7	8	9
result	7			6	5	3	3	3		1
					↑					

Construct result

priority	3	6	1	3	3	7	5	6	1	6
								↑		
count index	0	1	2	3	4	5	6	7		
count value	10	9	8	5	5	4	2	0		
							↑			
result index	0	1	2	3	4	5	6	7	8	9
result	7		6	6	5	3	3	3		1
			↑							

Construct result

priority	3	6	1	3	3	7	5	6	1	6
									↑	
count index	0	1	2	3	4	5	6	7		
count value	10	8	8	5	5	4	2	0		
		↑								
result index	0	1	2	3	4	5	6	7	8	9
result	7		6	6	5	3	3	3	1	1
									↑	

Construct result

priority	3	6	1	3	3	7	5	6	1	6
										↑
count index	0	1	2	3	4	5	6	7		
count value	10	8	8	5	5	4	1	0		
							↑			
result index	0	1	2	3	4	5	6	7	8	9
result	7	6	6	6	5	3	3	3	1	1
		↑								

Result

priority	3	6	1	3	3	7	5	6	1	6	
											↑
count index	0	1	2	3	4	5	6	7			
count value	10	8	8	5	5	4	1	0			
↑											
result index	0	1	2	3	4	5	6	7	8	9	
result	7	6	6	6	5	3	3	3	1	1	
↑											